

E_{11}, K_{11} and EE_{11}

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Abstract

In the study of conjecture on M-theory as a non-linear realization E_{11}/K_{11} we present arguments for the following: 1) roots of K_{11} coincide with the roots of Kac-Moody algebra EE_{11} with Dynkin diagram given in the paper, 2) one of the fundamental weights of EE_{11} coincides with l_1 weight of E_{11} , known to contain 11d supergravity brane charges. The statement 1) is extended on E_{10} and E_9 algebras.

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1 Introduction

The search for appropriate description of superstrings/M theory remains one of the important theoretical goals. It was conjectured [1, 2, 3, 4] that nonlinear realization of E_{11} Kac-Moody algebra (group) can be relevant for M-theory. $E_n, n \leq 10$ groups appeared previously as U-duality symmetries of dimensionally reduced M-theory, and E_{10} has been suggested as a candidate for possible description of M-theory [5, 6, 7]. The E_{11} conjecture join the E_n symmetry, usually linked to dimensionally reduced theory, and full multi-dimensional dependence of (infinitely many) fields of nonlinear realization sigma-model E_{11}/K_{11} . There is no precise form of the conjecture, and some important ingredients of the theory, e.g. description of space-time, fermionic content and supersymmetry relations remain unclear. Connection of this hypothesis to some other lines of search of hidden symmetries of M-theory, particularly to $OSp(1 | 64)$ conjecture [8] is also unclear. As mentioned in [9], corresponding even generators appear at low levels of E_{11} , but their commutators do not close into $Sp(64)$. In the present paper we would like to contribute to clarification of some group-theory aspects of E_{11}/K_{11} nonlinear model, and present some observations, which can help in inclusion of supersymmetry in the model. Our main attention is the gauged subgroup K_{11} . It is described as a subgroup of E_{11} , invariant w.r.t. the Chevalley involution (cf. Section 2). It is not described as a Kac-Moody algebra, and one of the problems we addressed to is to clarify how close is it to some Kac-Moody algebra. The first step, in Section 2, is to find the rank of K_{11} , and it was shown to be 11, as implied by its notation. In Section 3 we suggest to consider the problem with respect to new Cartan basis, assuming, in analogy with finite-dimensional simple groups, that all Kac-Moody defining relations have the same form in the new basis. We suggest a definition of K_{11} in a new basis, identify some Kac-Moody subalgebra of K_{11} , which we call EE_{11} , with Dynkin diagram at (17) and argue that roots of K_{11} coincide with those of EE_{11} . This statement is not proved, we only have been able to check that up to level 146 of group E_{11} . If it is correct, then generators of these two groups, corresponding to the real roots, completely coincide, because real roots have multiplicity one. Difference can arise in the sector of imaginary roots only. Similar statements are right for E_9 and E_{10} , the Dynkin diagrams of corresponding subalgebras are given at (21), (19). Situation for E_9 seems to be even more advanced, because both E_9 and $D_9^{(1)}$ (analog of EE_{11}) are of affine Kac-Moody type, for which multiplicities are known to be equal to

the rank of the algebra, so, conclusion seems to be that compact subgroup of E_9 is Kac-Moody algebra with Dynkin diagram given in (21).

It is worth mentioning that we carried out similar calculations for a few other groups, including E_{12} , and didn't find similar phenomena.

In Section 4 we consider representations of EE_{11} and E_{11} . We show that one of the fundamental weights of EE_{11} coincides with the l_1 fundamental weight of E_{11} , which is identified by West [10] as containing all 11d branes charges appearing in the r.h.s. of 11d susy relation. Perhaps this statement can be a hint for generalization of 11d susy relation on E_{11} . Moreover, it may appear to be a generalization of the observation of de Wit-Nicolai [11], that, depending on dimensionality, some charges in the r.h.s. of susy relation join into representations of corresponding E_n duality groups.

Conclusion contains discussion of present results and possible ways of further developments.

2 Rank of K_{11}

The non-linear realization suggested in [1] has E_{11} as a (global) Kac-Moody group, factorized over (local) group K_{11} (here and below we, following [1], are not making difference in notations between groups and algebras). The Lie algebra of E_{11} is a particular case of Kac-Moody [12] Lie algebra with basic generators h_i, e_i, f_i and relations:

$$[h_i, e_j] = A_{ij}e_j \tag{1}$$

$$[h_i, f_j] = -A_{ij}f_j \tag{2}$$

$$[e_i, f_j] = \delta_{ij}h_j \tag{3}$$

$$ad(e_i)^{(1-A_{ij})}e_j = 0, \tag{4}$$

$$ad(f_i)^{(1-A_{ij})}f_j = 0 \tag{5}$$

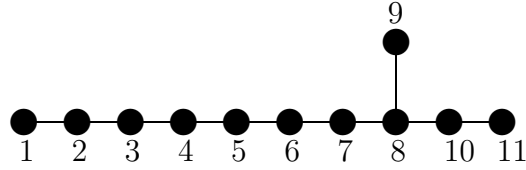
All other generators should be obtained from these generators by all possible multiple commutators, factorized over relations (1)-(5). In these expressions A_{ij} is the Cartan matrix, which completely characterizes the algebra, and which we assume symmetrizable, with 2 on main diagonal, and some negative integers off diagonal, with additional requirement that if some off-diagonal element is non-zero, then its transposed one is non-zero, too.

Actually in the present work only symmetric Cartan matrix will appear, with non-zero off-diagonal elements all equal (-1).

Most important for us among them is E_{11} Cartan matrix, which is given below:

$$E_{11} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \quad (6)$$

Alternatively it can be given by Dynkin diagram:



Dynkin diagram of E_{11} algebra (7)

Subalgebra K_{11} consists of generators invariant under Chevalley involution:

$$h_i \rightarrow -h_i \quad (8)$$

$$e_i \rightarrow -f_i \quad (9)$$

$$f_i \rightarrow -e_i \quad (10)$$

$$(11)$$

These definitions evidently extend to any Kac-Moody algebra. We shall call subalgebras K_n the Chevalley-invariant subalgebras, or compact subalgebras. As noted in [3], [9], some additional signs ϵ_i should be introduced in

r.h.s. of above expressions to take into account possibility of different signatures of space-time, and they correspond to different real forms of K_{11} . Below we shall be interested in problems, which are independent from the field of coefficients, namely in the rank of K_n and its possible Dynkin diagram, so we shall assume the field of coefficients to be the complex numbers. We assume that all our results will have some counterpart for different real forms of E_n and K_n , corresponding to different compactifications of 11d theory (see [9, 13, 14]). As is well-known, in a finite-dimensional case real forms all are based on the same Dynkin diagrams, decorated to become Vogan or Satake diagrams [15, 16], so by expecting something similar for Kac-Moody algebras, we ascertain the importance of understanding of the relations between K_n and Kac-Moody algebras with definite Dynkin diagrams.

For a finite-dimensional E_n groups there are following groups K_n (table from [1], see also [11]):

D	E_n	K_n	
11	1	1	
10, <i>IIB</i>	$SL(2)$	$SO(2)$	
10, <i>IIA</i>	$SO(1,1)/Z_2$	1	
9	$GL(2)$	$SO(2)$	
8	$E_3 \sim SL(5) \times SL(2)$	$U(2)$	(12)
7	$E_4 \sim SL(5)$	$USp(4)$	
6	$E_5 \sim SO(5,5)$	$USp(4) \times USp(4)$	
5	E_6	$USp(8)$	
4	E_7	$SU(8)$	
3	E_8	$SO(16)$	

We see that the rank of group K can differ from (be less than) that of E . We choose the following way of finding the rank of K_n . Consider the subgroup of E_n , generated by Dynkin diagram of E_n , taken without rightmost node (such subgroup for E_{10} was recently discussed in [17]). That is a Dynkin diagram of $SO(2n-2)$ group. We shall use the fact that Chevalley-invariant subgroup of $SO(2n)$ is $SO(n) \times SO(n)$. For E_{11} this means that we can find 10 Cartan generators of $SO(20)$ which belong to its Chevalley-invariant subgroup $SO(10) \times SO(10)$. It remains to find one additional invariant generator, commuting with these 10 to prove that rank of Chevalley-invariant subgroup of E_{11} is 11. With that purpose we decompose the E_{11} algebra w.r.t. the $SO(20)$. Decomposition is naturally graded by the powers of generators e_{11}

and f_{11} . Remind that the space of generators consists of positive and negative roots generators, constructed by powers (commutators) of solely e and f generators, respectively. From relations (1)-(5) one sees that subspace with given power of e_{11} and f_{11} is invariant w.r.t. the $SO(20)$. So, if we find a generator constructed from e -s and commuting with Cartan's of $SO(10) \times SO(10)$, we can add or subtract (depending on overall power of e -s) a similar generator constructed from f -s, and obtain the Chevalley-invariant generator, commuting with first 10 Cartan generators. Since it is anti-Hermitian, under appropriate definition of hermiticity, we assume that its action will be semi-simple, so it can be considered as 11-th Cartan generator.

We can change by $SO(20)$ rotation the basis of Cartan generators from initial one, with properties (8) under involution, to new one, so that its ten Cartan generators will be those of $SO(10) \times SO(10)$, which are involution-invariant Cartan generators, and we can calculate the roots of E_{11} in the new basis, and look for the roots which have zeros at first 10 positions of its weight vector. That will mean that commutator of corresponding generator(s) with first 10 Cartan generators is zero and so we have complete set of invariant Cartan generators. According to [16], zero weights appear in representations with highest weight expressible through roots with integer coefficients. Since in Kac-Moody algebras number of representations is infinite, it is "very likely" that such representation exists. For finite-dimensional algebras it may not be the case.

We use computer to generate roots of E_{11} and indeed the requested root is found on level (i.e. height in E_{11}) 47. It is in 4-th rank antisymmetric tensor representation of $SO(20)$, which can be seen from the corresponding highest weight. The root found, its weight, highest root of the module and the weight of highest root are the following:

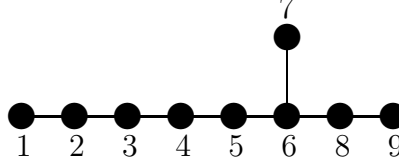
$$\begin{aligned} \text{root} &= (1, 2, 3, 4, 5, 6, 7, 8, 4, 5, 2) \\ \text{weight} &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1) \\ \text{highest root} &= (2, 4, 6, 8, 9, 10, 11, 12, 6, 7, 2) \\ \text{highest weight} &= (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, -3) \end{aligned}$$

Note that power of 11-th root is 2. The roots linear by 11-th prime root combine into spinorial representation of $SO(20)$, which does not have vectors with zero weights, so the new Cartan generator appeared immediately on the first possible level, namely that of second power of e_{11} . This root is imaginary, with square -2, so it is possible that number of corresponding generators is greater than one. Of course that will not mean that we have found more than 11 Cartan generators, since they may not commute with

each other.

One can repeat this calculation for different E_n groups, with some subtleties for n even.

For the E_9 group, with Dynkin diagram



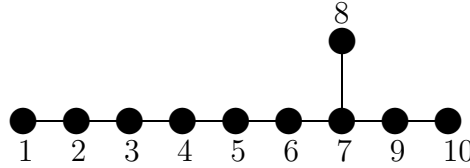
Dynkin diagram of E_9 algebra (13)

In the case of E_9 if we take out the rightmost root we will be left with $SO(16)$ diagram which has compact sub-algebra $SO(8) \times SO(8)$ with rank 8. So to prove that Chevalley-invariant sub-algebra of E_9 has rank 9 we have to find a generator which commutes with all Cartan generators of $SO(16)$. In analogy with E_{11} above, we see that there exists a root that has zeros on first 8 positions of its weight vector, hence commutes with all Cartan generators of $SO(16)$. That root is found on level 30 (and actually is the only one on that level) and has the form:

(1, 2, 3, 4, 5, 6, 3, 4, 2)

This proves that compact sub-algebra of E_9 has rank 9.

E_{10} has a following Dynkin diagram:



Dynkin diagram of E_{10} algebra (14)

For E_{10} we take out the second root from the left (i.e. root number two), remaining diagram is that of $SO(3) \times E_8$. Its compact subalgebra is $SO(2) \times SO(16)$ and has the same rank 9. So, as for E_{11} , we can change to new invariant Cartan basis and look for a root with 0 at first and last 8 positions

in its weight vector. Then corresponding generator(s) would commute with all Cartan generators and the statement that compact sub-algebra of E_{10} has rank 10 will be proved. Indeed such root exists on level 61 and has the following coefficients in its decomposition over simple roots:

$$(1, 2, 4, 6, 8, 10, 12, 6, 8, 4)$$

Module of this root is -2, scalar products with simple roots (i.e. components of its weight) are (0, -1, 0, 0, 0, 0, 0, 0, 0, 0).

3 K_n in a new basis

In a finite-dimensional simple Lie algebras one can consider different Cartan bases and they all are equivalent (in a complex field of coefficients), particularly they are connected through group automorphism. We are not aware of a similar general statement for Kac-Moody algebras (although see note at p.141 of [12]), and will accept as a hypothesis that for E_{11} we can change to the Cartan basis, introduced in the previous Section, and in this new basis it will be described by the same expressions (1)-(5), where generators e and f are new generators, constructed from previous ones (obtained by group automorphism, in the finite-dimensional analogue). The next question is: what will Chevalley involution (8) look like in the new basis. Since it commutes with new Cartan generators and can be simultaneously diagonalized, we can assume that its action on the new simple generators is multiplication by plus or minus one. Actually this statement is not correct generally, for any involution, as is known from finite-dimensional theory - some automorphisms of Dynkin diagram can be involved, which give rise to additional lines in Vogan (or Satake) diagrams [15, 16], namely the lines indicating order 2 orbits of involution on the set of simple roots. Note that our maximally compact basis of Cartan generators corresponds to Vogan type diagrams. But for appearance of such an automorphisms the Dynkin diagram should have corresponding property, moreover, for Chevalley involution, at finite dimensions, particularly for E_7, E_8, D_{4n} , i.e. for algebras with maximal rank of K_n subgroup, Chevalley involution is represented by multiplication by minus one of all simple generators. We shall assume the same behavior for E_{11} . This is second crucial assumption of the present paper.

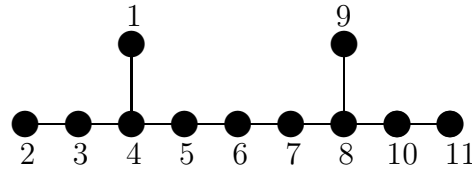
Evidently, with this assumption, in the new basis, generators of K_{11} can be described as all even generators of E_{11} , and their roots are all even roots of E_{11} .

Now we shall try to construct these roots from some basic even roots. We would like to introduce the following subalgebra of E_{11} : take the roots of following generators (Lie algebra commutators are implied)

$$\begin{aligned} a_1 &= e_7 e_8 e_9 e_{10}, a_2 = e_1 e_2, a_3 = e_3 e_4, a_4 = e_5 e_6, \\ a_5 &= e_7 e_8, a_6 = e_{10} e_{11}, a_7 = e_8 e_9, a_8 = e_6 e_7, \\ a_9 &= e_8 e_{10}, a_{10} = e_4 e_5, a_{11} = e_2 e_3, \end{aligned} \tag{15}$$

Definition of a_1 is actually unique, up to overall sign, since although Lie brackets can be arranged in different ways, results coincide. Then one can find corresponding Cartan matrix and Dynkin diagram:

$$\begin{array}{cccccccccccc} 2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \end{array} \tag{16}$$



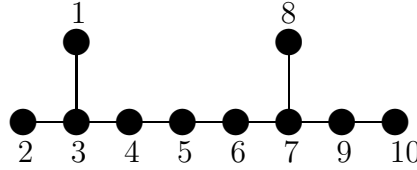
Dynkin diagram of EE_{11} algebra (17)

where simple roots in (17) are enumerated in agreement with (16). One can construct the corresponding abstract Kac-Moody algebra, we denote it by EE_{11} since it contains two E type tails, and this notation is similar

to that of hyperbolic algebras - AE, BE, CE, DE. All roots (15) are real, hence according to [18], algebra (17) is isomorphic to subalgebra in E_{11} , generated by generators (15). Our hypothesis is that perhaps this algebra has the same roots as K_{11} , i.e. all even roots of E_{11} . Statement seems to be simple, nevertheless, we were not able to prove it algebraically, due to unknown structure of roots system. Instead we checked that up to level 146 by the help of computer program, which generates the roots for an arbitrary input Dynkin diagram. The number of roots up to the level 146 (inclusively) is 19661788 (without counting multiplicity), so coincidence is considerable. Since multiplicity of real roots is one, this statement means that these two algebras coincide at least in a sector of real roots. It is not clear whether such coincidence exists also for imaginary roots (that would mean that groups K_{11} and EE_{11} coincide), at least imaginary root generators of EE_{11} belong to K_{11} .

The similar statements are correct for E_{10} and E_9 groups. Corresponding composite roots (generators) and Dynkin diagram for E_{10} are:

$$\begin{aligned} a_1 &= e_6 e_7 e_8 e_9, a_2 = e_2 e_3, a_3 = e_4 e_5, a_4 = e_6 e_7, a_5 = e_9 e_{10}, \\ a_6 &= e_7 e_8, a_7 = e_5 e_6, a_8 = e_7 e_9, a_9 = e_3 e_4, a_{10} = e_1 e_2 \end{aligned} \quad (18)$$

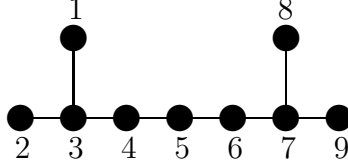


Dynkin diagram of DE_{10} algebra (19)

For E_{10} we check our hypothesis up to level 200, the number of roots up to that level is 18441198, and no deviation is found.

For E_9 :

$$\begin{aligned} a_1 &= e_1 e_2, a_2 = e_5 e_6 e_7 e_8, a_3 = e_3 e_4, a_4 = e_5 e_6, \\ a_5 &= e_8 e_9, a_6 = e_6 e_7, a_7 = e_4 e_5, a_8 = e_6 e_8, a_9 = e_2 e_3 \end{aligned} \quad (20)$$



Dynkin diagram of $D_9^{(1)}$ algebra (21)

This affine E_9 case is easy to check by computer, due to linear (not exponential, as in other, non-affine algebras) growth of number of roots. We check situation up to the level 10000, with complete confirmation of coincidence of roots of $D_9^{(1)}$ and even roots of E_9 . Here we can make a conclusion on complete coincidence, including imaginary roots, since both algebras - $D_9^{(1)}$ and E_9 are affine Kac-Moody algebras, multiplicity of imaginary roots of which depends on the rank only, hence not only even roots, but even generators of E_9 coincide with all generators of $D_9^{(1)}$. This conclusion disagrees with the statement of [19]. It may mean that some of our assumptions are incorrect: either a possibility of changing to the new, invariant Cartan basis, or representation of Chevalley involution in new basis as reflection of all simple generators.

One can provide an algebraic proof that all even generators of E_9 coincide with all generators of $D_9^{(1)}$. This is due to the fact, that the set of all roots of the (untwisted) affine algebra is completely known (see convenient physical exposition in [20]). Namely, that is the set of all non-zero triplets $(\alpha, 0, n)$, where α are all roots of finite algebra, on which affine one is based, including zero. Positive roots are those with $n > 0$, and, at $n = 0$, roots $(\alpha_+, 0, 0)$, with positive roots at first place. The set of simple roots is $(\alpha_i, 0, 0)$ (corresponding to the roots of finite algebra) and (additional affine root) $(-\psi, 0, 1)$, where ψ is maximal positive root of finite algebra. The set of even roots is $(\alpha, 0, n)$ with even α . We can take the set of all positive roots of $D_9^{(1)}$, substitute the expressions (20) of roots of $D_9^{(1)}$ through the roots of E_9 and check that we obtain in this way all positive even roots of E_9 . Denote all roots of D_8 by $\beta = \sum_{k=2}^9 m_k \beta_k$ where $\beta_k, k = 2 \div 9$ are simple roots, and set m_k here and below exactly parameterize all roots of D_8 . We denote simple affine roots β_1 for $D_9^{(1)}$ and α_1 for E_9 and use the same notation $\beta_k, k = 2 \div 9$ for remaining

simple roots $(\beta_i, 0, 0)$ of $D_9^{(1)}$. So, expanding an arbitrary $D_9^{(1)}$ root over simple roots, we have:

$$(\beta, 0, n) = \left(\sum_{k=2}^9 m_k \beta_k, 0, n \right) = \sum_{k=1}^9 n_k \beta_k \quad (22)$$

$$n_1 = n, n_k = n n_k^\psi + m_k \quad (23)$$

where n_k^ψ are coefficients of expansion of highest root:

$$\psi = \sum_{k=2}^9 n_k^\psi \beta_k \quad (24)$$

Substituting expressions (20) for $D_9^{(1)}$ roots β through roots α of E_9 , we obtain

$$(\beta, 0, n) = (n(-\lambda + \alpha_2) + \sum_{k=2}^9 n n_k^\psi \beta_k(\alpha) + \sum_{k=2}^9 m_k \beta_k(\alpha), 0, n) \quad (25)$$

where λ is highest positive root of E_8 . The sum of first two terms in (25) is zero:

$$n(-\lambda + \alpha_2) + \sum_{k=2}^9 n n_k^\psi \beta_k(\alpha) \quad (26)$$

which can be easily checked with the help of (20) and components of highest roots:

$$\psi \sim (12222211) \quad (27)$$

$$\lambda \sim (23456342) \quad (28)$$

Final form of equation (25)

$$(\beta, 0, n) = \left(\sum_{k=2}^9 m_k \beta_k(\alpha), 0, n \right) \quad (29)$$

is exactly the statement that even levels of E_8 coincide with group $SO(16)$. This statement is the finite dimensional analog and basis of our statements on Kac-Moody algebras $E_n, n = 11, 10, 9$.

4 On a representations of EE_{11}

One can easily obtain the weights of the highest weight vectors of fundamental representations of E_{11} . These are the rows of an inverse Cartan matrix:

$$EE_{11}^{-1} = \frac{1}{8} \begin{bmatrix} -1 & 2 & 4 & 6 & 9 & 12 & 15 & 18 & 9 & 12 & 6 \\ 2 & -4 & 0 & 4 & 6 & 8 & 10 & 12 & 6 & 8 & 4 \\ 4 & 0 & 0 & 8 & 12 & 16 & 20 & 24 & 12 & 16 & 8 \\ 6 & 4 & 8 & 12 & 18 & 24 & 30 & 36 & 18 & 24 & 12 \\ 9 & 6 & 12 & 18 & 15 & 20 & 25 & 30 & 15 & 20 & 10 \\ 12 & 8 & 16 & 24 & 20 & 16 & 20 & 24 & 12 & 16 & 8 \\ 15 & 10 & 20 & 30 & 25 & 20 & 15 & 18 & 9 & 12 & 6 \\ 18 & 12 & 24 & 36 & 30 & 24 & 18 & 12 & 6 & 8 & 4 \\ 9 & 6 & 12 & 18 & 15 & 12 & 9 & 6 & -1 & 4 & 2 \\ 12 & 8 & 16 & 24 & 20 & 16 & 12 & 8 & 4 & 0 & 0 \\ 6 & 4 & 8 & 12 & 10 & 8 & 6 & 4 & 2 & 0 & -4 \end{bmatrix} \quad (30)$$

We would like to compare these weights with those of E_{11} :

$$E_{11}^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 3 & 4 & 2 \\ 0 & 0 & 2 & 4 & 6 & 8 & 10 & 12 & 6 & 8 & 4 \\ 1 & 2 & 3 & 6 & 9 & 12 & 15 & 18 & 9 & 12 & 6 \\ 2 & 4 & 6 & 8 & 12 & 16 & 20 & 24 & 12 & 16 & 8 \\ 3 & 6 & 9 & 12 & 15 & 20 & 25 & 30 & 15 & 20 & 10 \\ 4 & 8 & 12 & 16 & 20 & 24 & 30 & 36 & 18 & 24 & 12 \\ 5 & 10 & 15 & 20 & 25 & 30 & 35 & 42 & 21 & 28 & 14 \\ 6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 24 & 32 & 16 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 11 & 16 & 8 \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 16 & 20 & 10 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 8 & 10 & 4 \end{bmatrix} \quad (31)$$

The only subtlety is that rows of inverse Cartan matrix express weights in a basis of simple roots of a given algebra, so for comparison we should express both in the same basis, e.g. in a basis of simple roots of E_{11} . The expression of simple roots of EE_{11} through the simple roots of E_{11} is given in (15). So we should multiply the matrix (30) from the right by the transformation matrix

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (32)$$

and obtain

$$(EE_{11})^{-1}T = \frac{1}{8} \begin{bmatrix} 2 & 8 & 10 & 16 & 18 & 24 & 26 & 32 & 14 & 20 & 12 \\ -4 & 0 & 4 & 8 & 12 & 16 & 20 & 24 & 12 & 16 & 8 \\ 0 & 8 & 8 & 16 & 24 & 32 & 40 & 48 & 24 & 32 & 16 \\ 4 & 16 & 20 & 32 & 36 & 48 & 60 & 72 & 36 & 48 & 24 \\ 6 & 16 & 22 & 32 & 38 & 48 & 54 & 64 & 34 & 44 & 20 \\ 8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 32 & 40 & 16 \\ 10 & 16 & 26 & 32 & 42 & 48 & 58 & 64 & 30 & 44 & 20 \\ 12 & 16 & 28 & 32 & 44 & 48 & 60 & 72 & 36 & 48 & 24 \\ 6 & 8 & 14 & 16 & 22 & 24 & 30 & 32 & 18 & 20 & 12 \\ 8 & 8 & 16 & 16 & 24 & 32 & 40 & 48 & 24 & 32 & 16 \\ 4 & 0 & 4 & 8 & 12 & 16 & 20 & 24 & 12 & 16 & 8 \end{bmatrix} \quad (33)$$

The second row of (33) and first row of (31) coincide. This is a very interesting coincidence, since exactly the first fundamental representation of E_{11} is shown, in [8], to contain the brane charges of 11d supergravity. The brane charges are appearing on the right hand side of the supersymmetry relation:

$$\begin{aligned} \{\bar{Q}, Q\} &= \Gamma^\mu P_\mu + \Gamma^{\mu\nu} Z_{\mu\nu} + \Gamma^{\mu\nu\lambda\rho\sigma} Z_{\mu\nu\lambda\rho\sigma}, \\ \mu, \nu, \dots &= 0, 1, 2, \dots, 10. \end{aligned} \quad (34)$$

This coincidence resembles the statement of [11], where it is noticed that at certain dimensions certain brane charges are not only in the representation of corresponding susy algebra's automorphism group, but also combine

into representations of U-duality group. For example, consider maximal susy algebra in 4 dimensions, obtained from reduction to 4d of 11d algebra (34). The corresponding 4d algebra has $SU(8)$ as an automorphism group. Scalar central charges appear from few sources: 7 from vector, 21 from membrane charge, and 28 from five-brane charge, altogether they combine into 56 of 4d U-duality group E_7 , which includes $SU(8)$ as its maximal compact subgroup. This and other similar statements of [11] is interesting to compare with above statement of coincidence of weights of E_{11} and EE_{11} , this is under investigation. Particularly, such a coincidence can reflect the possibility of extension of corresponding representation of EE_{11} to the l_1 representation of E_{11} .

5 Conclusion

In the present paper we consider the E_n and K_n groups, involved in different symmetry hypothesis of M-theory, particularly non-linear realization, at $n=11, 10, 9$. We make few hypothesis, extending some facts from finite group theory to Kac-Moody case. These crucial assumptions are first, possibility to change to new basis of Cartan generators, and, second, representation of Chevalley involution as reflection (multiplication by (-1)) of all simple roots generators. Both statements are supported in finite-dimensional simple Lie algebras, but are not clear for Kac-Moody algebras. With this assumption we present an observation, based on computer check up to some high levels of roots, that for E_{11} , E_{10} and E_9 algebras roots of compact (Chevalley-invariant) subalgebra coincide with roots of some Kac-Moody algebra, corresponding Dynkin diagrams are given by (17), (19) and (21) correspondingly.

As noticed in body of paper, above result means that generators, corresponding to real roots, coincide, since they are unique. For further study of hypothesis mentioned, one has to calculate the multiplicities of imaginary roots and compare them.

The naturally arising problem is to reformulate in a new basis the known properties of E/K nonlinear model, e.g. the supergravity solutions. In [4] they have been represented purely in the group language, so it seems to be a reasonable task. It may appear that some new solutions will naturally appear in a new basis. Another important question is to find an image of A_{10} subgroup, which serves in [1, 2, 8] to purpose of finding a contact with usual formulations of supergravities.

Besides, the new basis has an advantage that it naturally leads to the

subgroups, which belong to compact subgroup K_{11} , and, particularly, naturally extend Lorentz group (which appears as a compact subgroup of A_{10} , roots 1-10 at (7). Different subgroups of K_{11} can be read off directly from Dynkin diagram (17) of its subgroup EE_{11} . Finally, it is worth to mention again the problem of finding a relations between l_1 representation of E_{11} and that of EE_{11} .

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